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Geodesic structure of standard static space–times

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Abstract

The geodesic structure of standard static space–times is studied and conditions are found which imply nonreturning and pseudoconvex geodesic systems. As a consequence, it is shown that if the Riemannian factor manifold F satisfies the nonreturning property and has a pseudoconvex geodesic system and if the warping function $f : F \to (0, \infty)$ is bounded above then the standard static space–time ${}_f(a, b) \times F$ is geodesically connected.

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1. Introduction

Warped product manifolds were introduced in general relativity as a method to find general solutions to Einstein's field equations [6,19]. Two important examples include generalized Robertson–Walker space–times and standard static space–times. The former are obviously a generalization of Robertson–Walker space–times and the latter a generalization of the Einstein static universe. In this paper we focus on the geodesic structure of standard static space–times.

We recall that a warped product can be defined as follows [6,19]. Let (B, g_B) and (F, g_F) be pseudo-Riemannian manifolds and also let $b : B \to (0, \infty)$ be a smooth function. Then the (singly) warped product, $B \times {}_bF$ is the product manifold $B \times F$ furnished with the

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metric tensor $g = g_B \oplus b^2 g_F$ defined by

$$g = \pi^*(g_B) \oplus (b \circ \pi)^2 \sigma^*(g_F),$$

where $\pi : B \times F \to B$ and $\sigma : B \times F \to F$ are the usual projection maps and * denotes the pull-back operator on tensors. A standard static space–time can be considered as a Lorentzian warped product where the warping function is defined on a Riemannian manifold called the fiber and acting on the negative definite metric on an open interval of real numbers, called the base. More precisely, a standard static space–time, ${}_f(a, b) \times F$ is a Lorentzian warped product furnished with the metric $g = -f^2\, \mathrm{d}t^2 \oplus g_F$, where (F, g_F) is a Riemannian manifold, $f : F \to (0, \infty)$ is smooth, and $-\infty \leq a < b \leq \infty$. In [19], it was shown that any static space–time is locally isometric to a standard static space–time.

Standard static space–times have been previously studied by many authors. Kobayashi and Obata [17] stated the geodesic equation for this class of space–times and the causal structure and geodesic completeness were considered in [2], where sufficient conditions on the warping function for nonspacelike geodesic completeness of the standard static space–time were obtained (see also [20]). In [1], conditions are found which guarantee that standard static space–times either satisfy or else fail to satisfy certain curvature conditions from general relativity. The existence of geodesics in standard static space–times have been studied by several authors. Sánchez [23] gives a good overview of geodesic connectedness in semi-Riemannian manifolds, including a discussion for standard static space–times. Results on the existence of geodesics in two-dimensional standard static space–times are stated in [12,13]. Additional references on geodesic connectedness and the existence of multiple geodesics and t-periodic geodesics in static space–times include [10,11,14,15,21,22].

Two of the most famous examples of standard static space–times are Minkowski space–time and the Einstein static universe [6,16] which is $\mathbb{R} \times \mathbb{S}^3$ equipped with the metric

$$g = -\mathrm{d}t^2 + (\mathrm{d}r^2 + \sin^2 r\, \mathrm{d}\theta^2 + \sin^2 r \sin^2\theta\, \mathrm{d}\phi^2),$$

where $\mathbb{S}^3$ is the usual three-dimensional Euclidean sphere and the warping function $f \equiv 1$. Another well-known example is the universal covering space of anti-de Sitter space–time, a standard static space–time of the form ${}_f\mathbb{R} \times \mathbb{H}^3$, where $\mathbb{H}^3$ is the three-dimensional hyperbolic space with constant negative sectional curvature and the warping function $f : \mathbb{H}^3 \to (0, \infty)$ defined as $f(r, \theta, \phi) = \cosh r$ [6,16]. Finally, we can also mention the exterior Schwarzschild space–time [6,16], a standard static space–time of the form ${}_f\mathbb{R} \times (2m, \infty) \times \mathbb{S}^2$, where $\mathbb{S}^2$ is the two-dimensional Euclidean sphere, the warping function $f : (2m, \infty) \times \mathbb{S}^2 \to (0, \infty)$ is given by $f(r, \theta, \phi) = \sqrt{1 - 2m/r}$, $r > 2m$ and the line element on $(2m, \infty) \times \mathbb{S}^2$ is

$$\mathrm{d}s^2 = \left(1 - \frac{2m}{r}\right)^{-1} \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2).$$

This paper is organized as follows. In Section 2 we give the formal definition of standard static space–times and state the geodesic equations for these space–times. In addition, we recall several definitions that are needed in later sections, including the definitions of a pseudoconvex geodesic system, and the disprisoning, unitrace and nonreturning properties for a manifold with linear connection. In Section 3 we present our results on the geodesic structure of standard static space–times. In Theorem 3.1 we show that the standard static space–time ${}_f(a, b) \times F$ satisfies the nonreturning property if the Riemannian factor F satisfies the nonreturning property. We state two different results which provide sufficient conditions for a standard static space–time to have a pseudoconvex geodesic system. In Theorem 3.2 we show that if the Riemannian manifold (F, g_F) is compact then ${}_f(a, b) \times F$ has a pseudoconvex geodesic system independent of the warping function f. In Theorem 3.3 we show that if (F, g_F) has a pseudoconvex geodesic system and the warping function f is bounded above then ${}_f(a, b) \times F$ has a pseudoconvex geodesic system. As a consequence of these results, we are able to show in Corollary 3.5 that if (F, g_F) has a pseudoconvex geodesic system and is nonreturning and if the warping function f is bounded above then ${}_f(a, b) \times F$ is geodesically connected. Geodesic connectedness is the property that arbitrary points of the space–time can be joined by a geodesic. Corollary 3.5 can be compared with the result in [10] in which it is shown that a standard static space–time is geodesically connected if the Riemannian manifold F is complete and the warping function satisfies $0 < \inf(f) \leqslant \sup(f) < \infty$. Corollary 3.5 stands as a peer with the theorem in [10] in that there are complete Riemannian manifolds which fail to be geodesically pseudoconvex and there are geodesically pseudoconvex and nonreturning Riemannian manifolds which are incomplete.

2. Preliminaries

In this section, we first give a formal definition of standard static space–times and state the geodesic equations for a standard static space–time.

Definition 2.1. Let (F, g_F) be a Riemannian manifold and also let $f : F \to (0, \infty)$ be a smooth map where $-\infty \leq a < b \leq \infty$. Then the standard static space–time $M = {}_f(a, b) \times F$ is the product of $((a, b), -\mathrm{d}t^2)$ and (F, g_F) furnished with the metric $g = -f^2\,\mathrm{d}t^2 \oplus g_F$.

Here, (F, g_F) is called the fiber and f is called the warping function. A standard static space–time is said to be trivial if the warping function f is constant.

Now we will recall the geodesic equations for a curve $\gamma = (\alpha, \beta) : I \to (a, b) \times F$ in a standard static space–time. These equations are special cases of the geodesic equations for a general warped product stated in [19].

Proposition 2.2. *A smooth curve $\gamma = (\alpha, \beta) : I \to (a, b) \times F$ in a standard static space–time of the form $M = {}_f(a, b) \times F$ with the metric $g = -f^2\,\mathrm{d}t^2 \oplus g_F$ is a geodesic if and only if the following hold*:

1. $$\alpha'' = \frac{-2}{f \circ \beta} \frac{\mathrm{d}(f \circ \beta)}{\mathrm{d}t} \alpha',$$
2. $$\beta'' = -(f \circ \beta)(\alpha')^2 \nabla^F(f)|_{\beta(t)},$$

where $\nabla^F(f)$ denotes for the gradient of f on F.

Remark 2.3. If $\gamma = (\alpha, \beta)$ is a geodesic in a standard static space–time, then $-(f \circ \beta)^4(\alpha')^2 \equiv C$ is constant (see page 208 of [19]) and $-(f \circ \beta)^2(\alpha')^2 + g_F(\beta', \beta') \equiv D$, i.e., the constant speed of the geodesic. Moreover, α turns out to be a pre-geodesic on (a, b) with the metric $-dt^2$.

The concept of a *pseudoconvex geodesic system*, and the *disprisoning*, *unitrace*, and *nonreturning* properties of a manifold with a linear connection have been used in a series of papers [3–5,7–9] to study the geodesic structure of such manifolds. In particular, results on the space of geodesics, geodesic connectedness, and Cartan–Hadamard type theorems have been established. We will be using these concepts in this paper and now will recall the definitions and briefly discuss some key facts. We state the definitions for an arbitrary space–time (M, g).

A space–time (M, g) is *geodesically pseudoconvex* if for each compact set K in M there is a compact set H such that each geodesic segment $\alpha : [a, b] \to M$, with $\alpha(a), \alpha(b) \in K$ satisfies $\alpha[a, b] \subseteq H$. The space–time (M, g) is *disprisoning* if for each inextendible geodesic $\alpha : (a, b) \to M$ and $t_0 \in (a, b)$ both the sets $\alpha(a, t_0]$ and $\alpha[t_0, b)$ fail to have compact closure. The space–time (M, g) is *unitrace* if for each p in M and each neighborhood of V of p there is a simple convex neighborhood U of p with $U \subseteq V$ such that any geodesic which enters U either leaves and never returns or retraces the same path every time it does return. Finally, the space–time (M, g) is *nonreturning* if each point p of M has arbitrarily small neighborhoods $U(p)$ such that any geodesic starting in $U(p)$ and leaving $U(p)$ fails to return to $U(p)$.

Clearly, if (M, g) satisfies the nonreturning property, then (M, g) also satisfies the disprisoning property [5] and the unitrace property [7]. In [4], it is shown that if (M, g) is both geodesically pseudoconvex and disprisoning and also (M, g) has no conjugate points, then (M, g) is geodesically connected and for each point $p \in M$ the map $\exp_p : T_p(M) \to M$ is a diffeomorphism and hence M is diffeomorphic to $\mathbb{R}^n$, where $n = \dim(M)$. In [5] it is shown that if (M, g) satisfies the nonreturning property then the property of geodesic pseudoconvexity is equivalent to the space of geodesics of (M, g) being Hausdorff. In [7] it is shown that if (M, g) is unitrace and the space of geodesics is Hausdorff then (M, g) is geodesically connected. Combining these results we see that if (M, g) is nonreturning and geodesically pseudoconvex then it is geodesically connected.

Beem has applied the above ideas in the setting of Lorentzian warped products (M, g) of the form $M = (a, b) \times_h F$ with the metric $g = -dt^2 \oplus h^2 g_F$, where $h : (a, b) \to (0, \infty)$ is smooth. These warped products are sometimes called generalized Robertson–Walker space–times [13]. In [9] it is proven that if (F, g_F) satisfies the nonreturning property and has a pseudoconvex geodesic system and there exist $t_1, t_2 \in (a, b)$ with $t_1 < t_2$ such that $h' \geq 0$ on (a, t_1) and $h' \leq 0$ on (t_2, b) then (i) $(a, b) \times_h F$ satisfies the nonreturning property and has a pseudoconvex geodesic system, (ii) the space of geodesics of $(a, b) \times_h F$ is a $2s$-dimensional Hausdorff manifold where $\dim(F) = s$, and (iii) $(a, b) \times_h F$ is geodesically connected.

3. Geodesic system

In this section, we consider the geodesic system of standard static space–times. We consider the disprisoning property and pseudoconvexity of standard static space–times and establish sufficient conditions for geodesic connectedness and for the space of geodesics to be a Hausdorff manifold.

Note that (a, b) with the metric $-\mathrm{d}t^2$ satisfies the nonreturning property and also it has a pseudoconvex geodesic system. We will apply techniques used in the proof of Theorem 3.2 of [9] and modify it by interchanging the roles of the basis and fiber to prove the following result.

Theorem 3.1. *Let* $M = {}_f(a, b) \times F$ *be a standard static space–time with the metric* $g = -f^2\,\mathrm{d}t^2 \oplus g_F$. *Then* (M, g) *satisfies the nonreturning property if and only if* (F, g_F) *satisfies the nonreturning property*.

Proof. It is known that standard static space–times are stably causal [2], and hence strongly causal. Thus, each point of ${}_f(a, b) \times F$ has sufficiently small neighborhoods such that any causal geodesic leaving such a neighborhood fails to return. Assume that the standard static space–time (M, g) does not satisfy the nonreturning property. Then there is some point (t_0, p_0) in M such that arbitrarily small neighborhoods exist containing this point and each such neighborhood has geodesics which leave and return. For sufficiently small neighborhoods, the geodesics that leave and return must be space-like.

Recall that geodesics in ${}_f(a, b) \times F$ project under π to pregeodesics of $((a, b), -\mathrm{d}t^2)$ but that they may possibly be constant geodesics of (a, b). By Proposition 2.2 and Remark 2.3, we see that space-like geodesics $\gamma = (\alpha, \beta)$ that have $\alpha'(s_0) = 0$ for some $s_0 \in (a, b)$ will have $\alpha'(s) \equiv 0$ and $\beta(s)$ will be a geodesic of (F, g_F). It is impossible that a geodesic γ of this type can leave arbitrarily small neighborhoods of M and return since then the geodesic β of (F, g_F) would leave arbitrarily small neighborhoods of F and return, thereby violating the nonreturning property of (F, g_F). Thus, any space-like geodesic $\gamma = (\alpha, \beta)$ which leaves arbitrarily small neighborhoods of (t_0, p_0) in M and returns must be of the form $\alpha'(s) \neq 0$ for all $s \in (a, b)$ and $\alpha'(s)$ will not change sign.

Suppose that $W(t_0, p_0)$ is a convex normal neighborhood of (t_0, p_0) in M containing space-like geodesics which leave $W(t_0, p_0)$ and return to $W(t_0, p_0)$. By the remarks in the previous paragraph, the projection of any such geodesic under π into (a, b) must be a nonconstant pregeodesic in $((a, b), -\mathrm{d}t^2)$. One can choose a convex normal neighborhood $U(t_0)$ in (a, b) such that any geodesic of $((a, b), -\mathrm{d}t^2)$ which leaves $U(t_0)$ must fail to return because of the nonreturning property of $((a, b), -\mathrm{d}t^2)$. Without loss of generality, we assume that $U(t_0) \subseteq \pi(W(t_0, p_0))$. Note that space-like geodesics of ${}_f(a, b) \times F$ starting in $W(t_0, p_0) \cap \pi^{-1}(U(t_0))$ project to pregeodesics which leave $U(t_0)$ and fail to return. Hence, the corresponding space-like geodesics starting in $W(t_0, p_0) \cap \pi^{-1}(U(t_0))$ may not return to $U(t_0) \times F$. This shows that ${}_f(a, b) \times F$ must be nonreturning.

The implication that if (M, g) is nonreturning then (F, g_F) is nonreturning follows easily from the observation that if β is a geodesic in (F, g_F) then, for a fixed $t_0 \in (a, b)$, $\gamma(t) = (t_0, \beta(t))$ is a geodesic in (M, g). □

Now we consider sufficient conditions to obtain geodesic pseudoconvexity of standard static space–times. The first theorem requires no condition on the warping function but a rather strong condition on the Riemannian manifold (F, g_F).

Theorem 3.2. *Let* $M = {}_f(a, b) \times F$ *be a standard static space–time with the metric* $g = -f^2\,\mathrm{d}t^2 \oplus g_F$. *If* (F, g_F) *is compact then the standard static space–time* (M, g) *has a pseudoconvex geodesic system*.

Proof. Let K be a compact set in M and let H be the compact set $\pi(K) \times F$. Suppose $\gamma = (\alpha, \beta) : [0, 1] \to M$ is an arbitrary geodesic in (M, g) such that $\gamma(0)$ and $\gamma(1)$ are in K. Then $\alpha : I \to (a, b)$ is a pregeodesic in $((a, b), -\mathrm{d}t^2)$ such that $\alpha(0)$ and $\alpha(1)$ are in $\pi(K)$. By Remark 2.3, $\alpha'(s)$ cannot change sign and hence, $\alpha[0, 1] \subseteq \pi(K)$. It follows that $\gamma[0, 1] \subseteq H$. □

In the case (F, g_F) is not compact but is pseudoconvex then ${}_f(a, b) \times F$ may fail to be pseudoconvex. For example, universal anti-de Sitter space–time ${}_f\mathbb{R} \times \mathbb{H}^3$ where $\mathbb{H}^3$ is the three-dimensional hyperbolic space and $f : \mathbb{H}^3 \to (0, \infty)$ is defined as $f(r, \theta, \phi) = \cosh r$ fails to be timelike pseudoconvex [8]. However, we can state the following.

Theorem 3.3. *Let $M = {}_f(a, b) \times F$ be a standard static space–time with the metric $g = -f^2\,\mathrm{d}t^2 \oplus g_F$. If (F, g_F) has a pseudoconvex geodesic system and $\sup(f) < \infty$ then the standard static space–time (M, g) has a pseudoconvex geodesic system.*

Proof. We will again modify the proof of Theorem 3.2 in [9] by interchanging the roles of the basis and fiber to conclude that nonspacelike geodesics satisfy pseudoconvexity as follows: let W be an arbitrary compact set in M and also let $\gamma = (\alpha, \beta) : [0, 1] \to M$ be a nonspacelike-like geodesic in (M, g) with endpoints $\gamma(0), \gamma(1)$ in W. Since $\alpha : [0, 1] \to (a, b)$ is a pregeodesic in $((a, b), -\mathrm{d}t^2)$ with endpoints $\alpha(0), \alpha(1)$ in the compact set $\pi(W)$ in (a, b) and by using the pseudoconvexity of $((a, b), -\mathrm{d}t^2)$, one can find a compact set U in (a, b) such that $\alpha[0, 1]$ is contained in U. Note that $V = \sigma(W)$ is compact in F, and hence $U \times V$ is compact in M. Every nonspacelike-like geodesic starting in $U \times V$ fails to return to $U \times V$ because if it did then the time function $T(t, p) = t$ for the standard static space–time (M, g) would not be monotonic along the nonspacelike-like geodesics. Hence, any nonspacelike geodesic with both endpoints in W must remain in the compact set $U \times V$. Now, we will consider the pseudoconvexity of space-like geodesics. Let $\gamma = (\alpha, \beta) : I \to (a, b) \times F$ be a spacelike geodesic where I is a closed and bounded interval. Then $g(\gamma', \gamma') \equiv D > 0$ on I and α is a pregeodesic and hence $(\alpha')^2 < k$ for some positive number $k > 0$ on I. Thus $g_F(\beta', \beta') < D + (\sup(f))^2 k^2 < \infty$ and this implies that β must be contained in a compact set in F because otherwise $g_F(\beta', \beta') = \infty$. Hence spacelike geodesics also satisfy pseudoconvexity. □

Example 3.4. Notice that in order to obtain the above result the assumption that (F, g_F) has a pseudoconvex geodesic system cannot be dropped, as the following example shows. Let M be the standard static space–time ${}_f(a, b) \times F$, where F is a Riemannian manifold chosen to be not pseudoconvex and f is arbitrary. To be specific, we can choose $(a, b) = \mathbb{R}$, F to be the punctured plane $\mathbb{R}^2 - (0, 0)$, and $f \equiv 1$. Let the compact set K be the (Euclidean) unit circle in the punctured plane centered at the origin and in the plane $t = 0$ of M. If we consider all spacelike geodesic segments with endpoints in K, they will include all curves of the form $\gamma(t) = (0, \beta(t))$, where $\beta(t)$ is a straight line geodesic segment in the punctured plane joining two points of K. No compact set in M contains all such geodesic segments and hence, M is not spacelike pseudoconvex.

As our final result, we state sufficient conditions for the space of geodesics $\mathcal{G}(M)$ [7] of a standard static space–time to be a Hausdorff manifold and to obtain geodesic connectedness. First recall that if an arbitrary n-dimensional manifold (M, g) is nonreturning then it is unitrace (see [7]) and also disprisoning (see [5]).

Corollary 3.5. *Let $M = {}_f(a, b) \times F$ be a standard static space–time with the metric $g = -f^2\,\mathrm{d}t^2 \oplus g_F$. Suppose that (F, g_F) satisfies the nonreturning property and has a pseudoconvex geodesic system. If $\sup(f) < \infty$ then*

1. *The space of geodesics of the standard static space–time (M, g), i.e., $\mathcal{G}(M)$ is a $2s$-dimensional Hausdorff manifold where $\dim(F) = s$.*
2. *The standard static space–time (M, g) is geodesically connected.*

Proof. First, it immediately follows from Theorems 3.1 and 3.3 that the standard static space–time (M, g) is both nonreturning and pseudoconvex and hence unitrace and disprisoning as mentioned above. Then by Corollary 5.6 in [5], the space of geodesics, $\mathcal{G}(M)$ is a $2s$-dimensional Hausdorff manifold where $\dim(F) = s$. Thus one can apply Theorem 4.2 in [7] to obtain the geodesic connectedness of (M, g). □

Remark 3.6. Using variational methods it has been shown [10] that if the Riemannian manifold F is complete and the warping function satisfies $0 < \inf(f) \leqslant \sup(f) < \infty$ then ${}_f(a, b) \times F$ is geodesically connected. In [18] it is shown that the condition $0 < \inf(f)$ can be dropped. There are Riemannian manifolds that are geodesically pseudoconvex and nonreturning yet fail to be complete. The simplest example is an open proper convex subset of $\mathbb{R}^n$ with the usual Euclidean metric. On the other hand, there are Riemannian manifolds which are complete yet fail to be geodesically pseudoconvex [3]. Hence, Corollary 3.5 is neither a special case of the result in [10] nor does it imply that result. Furthermore, the imposed conditions in [10,18] imply that the space–times are globally hyperbolic yet the hypotheses in Corollary 3.5 do not imply global hyperbolicity.

Remark 3.7. The fact that there are Riemannian manifolds which are complete and not geodesically pseudoconvex together with the result from [10] referred to in Remark 3.6 shows that the sufficient conditions for geodesic connectedness in Corollary 3.5 are not necessary conditions.

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